Answers and solutions to problems of the Red Difficulty Level MathCat

Problem 1. (5 credits) The digit 5 was added to the right of a natural number N, and the result of this operation it 1823 greater than N. Find N.
Answer: 202
Solution: Adding a digit to the end of the number N is the same as multiplying N by 10 and adding that digit. Therefore, $10N + 5 = N + 1823$, i.e. $N = 202$.

Problem 2. (6 credits) A student group consists of 25 students with certain math, physics, and chemistry scores. 8 students have the same math and physics scores. 13 students have different math and chemistry scores. Find the minimum possible number of students whose physics and chemistry scores are different.
Answer: 4
Solution: 12 students have the same math and chemistry scores. A maximum of 8 of them can have the same physics scores as well; this means that all three scores of at least 4 students are not the same. This happens if, for example, 8 students have the same scores in all three subjects, another 4 students have the same math and chemistry scores unlike their physics scores, and the remaining 13 students have the same physics and chemistry scores unlike their math scores.

Problem 3. (8 credits) There are 29 coins of denominations of 1, 2, …, 29 cents on a table. Tom Sawyer and Huckleberry Finn take turns taking three coins from the table. When two coins remain on the table, the first player wins a prize equal to the difference in their denominations. What is the maximum prize this player can win, regardless of his opponent's play?
Answer: 16
Solution: Let's show that the first player can always get a difference of 16. On his first move, he takes coins 14, 15, and 16, and then groups all the remaining numbers on coins into pairs with a difference of 16. On each move, the second player either takes three numbers from different pairs or takes any entire pair and one number from another pair. In the first case, the first player takes all the remaining numbers from these pairs, and in the second case, he takes one remaining number and any entire pair. At the end, the numbers remaining will be from one pair, and this means that the difference will be 16.
Now let's show that the second player can prevent the first player from getting more than 16. He can take three coins of the highest denomination each time. During his 4 moves, he can take 12 coins; this means that the coins of denominations from 18 to 29 cents will be absent on the table in the end, and the maximum remaining difference between 1 and 17 is exactly 16.

Problem 4. (8 credits) The point E on the side BC of the parallelogram $ABCD$ is such that the lines AE and BD are perpendicular. The point F on the side AD is equidistant from the vertices B and D. $AF = 5$, and $BE = 4$. Find BF.
Answer: 9
Solution: Let G be a point on the side BC equidistant from B and D. This means that FG is the perpendicular bisector of the diagonal of the parallelogram, i.e. $FG \parallel AE$. Therefore, the quadrangle $AEGF$ is a parallelogram, and $EG = AF = 5$. This means that $BG = 5 + 4 = 9$, and since $BFDG$ is a rhombus, $BF = BG$.

Problem 5. (10 credits) Robin the Bobbin eats candies every day. Over any 3 consecutive days, he eats no more than 10 candies, and over any 5 consecutive days, he eats no more than 18 candies. What's the largest amount of candies he can eat in 19 days?
Answer: 66
Solution: Let's split all 19 days into three groups of three days and two groups of five days. Robin the Bobbin eats no more than 18*2+3*10=66 candies during the whole 19-day period. There's a case for exactly 66 candies: he eats 6 candies on the first day, 2 candies on the second day, and 2 candies on the third day, and then he repeats this three-day sequence: 6,2,2,6,2,2,…6.

Problem 6. (10 credits) A lazy mouse and a piece of cheese are on a grid of hexagonal cells, as shown in Figure 1. The mouse can move from its cell to any adjacent cell during its turn. However, it:
1) never moves to a cell that is higher than its cell,
2) never returns to any previous cell.
How many ways does the mouse have to reach the cheese?
Answer: 384
Solution: Note that for any two cells within a single horizontal row, there is only one way from the first cell to the second one. Therefore, knowing the cells where the mouse moves to another row, we can uniquely reconstruct

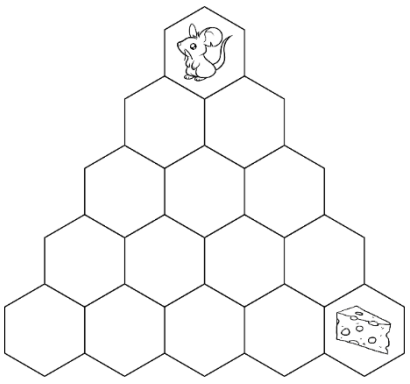

Figure 1

its route. If we number the rows starting with 1, the number of ways to descend from the row k to the next one will be $2k$, since the k-th row has k cells, and each such cell shares two sides with cells of the next row. Therefore, there are $2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n-2) = 2^{(n-1)} \cdot (n-1)!$ ways to reach any cell in the n-th row. After substituting $n = 5$, we find the answer: $2^4 \cdot 4! = 16 \cdot 24 = 384$.

Problem 7. (12 credits) Alex, Boris, and Victor signed up for a 1'000-meter race. All runners started at the same time; Victor and Boris started at the same end of the track, and Alex started at the opposite end. By the time Alex met Victor, the distance run by him was 20% less than Boris's remaining distance. When Alex reached 400 meter point, Victor hasn't finished yet. What was the distance between Boris and Victor at that time?
Answer: 100
Solution: According to the conditions, when Alex and Victor met, the distance between Boris and Victor was 20% of Boris's remaining distance, and the distance run by Alex was 80% of Boris's remaining distance. This means that Alex's speed is four times greater than the difference between the speeds of Victor and Boris, which gives 400:4=100 meters.

Problem 8. (13 credits) Tweedledum wrote down an arithmetic progression of three integers, Tweedledee added 2 to the second number and 12 to the third number, and the brothers finally got a geometric progression with a non-zero integer denominator. What could this denominator be?
Answer: -1; 2; 3
Solution: Let a be the first term of the progression, d be the step of the arithmetic progression, q be the denominator of the geometric progression. In this case, $aq = (a + d + 2)$ and $aq^2 = (a + 2d + 12)$. After subtracting the first equation multiplied by two from the second equation, we get $aq^2 - 2aq = (a + 2d + 12) - 2(a + d + 2)$, that is $a(q-1)^2 = 8$. This means that $(q-1)^2$ is a divisor of 8, which is only possible if $(q-1)^2 = 1$ or $(q-1)^2 = 4$. Therefore, $q \in \{-1; 0; 2; 3\}$. Sequences for each q:
$q = -1; a = 2; d = -6$: arithmetic progression $2; -4; -10$, geometric progression $2; -2; 2$.

$q = 2; a = 8; d = 6$: arithmetic progression $8; 14; 20$, geometric progression $8; 16; 32$.

$q = 3; a = 2; d = 2$: arithmetic progression $2; 4; 6$, geometric progression $2; 6; 18$

Problem 9. (13 credits) Each square on a 5×5 board contains either the letter D or the letter C. Such letter is chosen randomly for each square, and the total number of the letters D in the row containing that square is added to the total number of B letters in the column containing that square. If this sum is even, the square is painted black; if it is odd, the square is painted white. Find the expected value of the number of black squares.
Answer: 12.5
Solution: At first, we choose any square A and then any square B that lies in the same row or column. We also change the color of the square A by replacing the letter in the square B with the opposite one. This means that for any square, the probability that it is black is $\frac{1}{2}$. The expected value of the number of black squares is $25 \cdot \frac{1}{2} = 12{,}5$.

Problem 10. (15 credits) Find all values of a, for which the equation $a = \sqrt[3]{x^2 - 6x + 7} + \sqrt[3]{x^2 - 2x - 1}$ has an odd number of solutions.
Answer: -2
Solution: After selecting the perfect squares $a = \sqrt[3]{(x-3)^2 - 2} + \sqrt[3]{(x-1)^2 - 2}$, we can see that the radicands transform into each other when x is replaced by $4 - x$, i.e. for each root, there is a root symmetrical with respect to the line $x = 2$. This means that if the number of the roots is odd, $x = 2$ is the root. After substituting this into the expression, we find that $a = -2$.