Answers and solutions to problems of the Yellow Difficulty Level MathCat

Problem 1. (6 credits) There are 14 people in a room. Each of them is either a knight (who makes only true statements) or a liar (who makes only false statements). Each one declared, "How awful! There are more liars than knights among those I see here!" How many liars are in the room?
Answer: 7
Solution: It follows from the liar's statement that no more than half of the 13 people are liars, i.e. the number of liars does not exceed 6 according to this statement. The liar making the statement must also be added, i.e. the total number of liars does not exceed 6+1=7.
It follows from the knight's statement that no less than half of the 13 people are liars, i.e. the number of liars is at least 7. Therefore, the total number of liars is 7.

Problem 2. (6 credits) All natural numbers having any two adjacent digits of different parity and the sum of their digits equal to 57 are written in ascending order. Which number is written second?
Answer: 8789898
Solution: First of all we have to find the first number in the sequence. The smallest number has at least 7 digits, since the sum of any 6 digits of such number is no more than 3*(8+9)=51. If such number has 7 digits, it is written in the following way: OEOEOEO or EOEOEOE (O=odd digit, E=even digit). In the first case, the sum of the digits is even, in the second one it is odd. We see that the EOEOEOE number is applicable. The greatest sum of the last six digits is 9+8+9+8+9+8=51, i.e. the smallest first digit is 57-51=6, and the first number in the series is 6989898. It follows from the above that only one 7-digit number begins with 6, and the next one begins with 8, and the smallest second digit is 7. Therefore, the second number in the series is 8789898.

Problem 3. (8 credits) Two teams of children shared some candies. The captain of the first team told that they had received 4/7 of all candies, and the captain of the second team told that they had received 1/2 of all candies. But suddenly it turned out that both captains thought that Wendy was on their team, and Peter was not. Wendy received 8 candies, and Peter received only 3 candies. How many candies were there in total?
Answer: 70
Solution: If the total number of candies is N, 4N/7+N/2 is the total number of candies in the case when Wendy is counted twice and Peter is not counted. Therefore, 4N/7+N/2=N+8-3, and N=70.

Problem 4. (8 credits) All squares of an 11x11 chessboard are painted white and black in a checkerboard pattern. All corners of the board are black. In how many ways can one red chip and one green chip be placed on the black squares being placed at the same time on the same horizontal or vertical line?
Answer: 560
Solution: There are 12 lines with 6 black cells and 10 lines with 5 black cells. In the first case, we have 6*5=30 options, in the second case we have 5*4=20 options. Total number of the ways: 12*30+10*20=560.

Problem 5. (10 credits) A five-digit number can be broken into a three-digit number and a two-digit number in two ways. In one of these cases, the sum of its parts is 168, in the other case the sum is 357. Find the initial five-digit number and specify all possible solutions.
Answer: 12345 and 32136
Решение: Let the unknown number be $\overline{AxB}$, where *A* and *B* are two-digit numbers and x is a digit.
Case 1: $\overline{Ax} + B = 10A + x + B = 168$, $A + \overline{xB} = 100x + A + B = 357$. When we subtract one equality from another, we get $11x = A + 21$. According to the first condition $A \leq 16$, therefore $x = 3$, $A = 12$.
In this case, $B = 45$.
Case 2: $\overline{Ax} + B = 10A + x + B = 357$, $A + \overline{xB} = 100x + A + B = 168$. When we subtract one equality from another, we get $11x = A - 21$. According to the first condition $A \leq 35$, therefore $x = 1$, $A = 32$.
In this case $B = 36$.

Problem 6. (10 credits) 14 gentlemen are members of a club. Over the course of a week, each member met other member once. Each time they met, the first gentleman asked the second member to pass on his greetings to all other members this second gentleman might see again. How many greetings were passed on in total?

Answer: 1092

Solution: After the first meeting, each gentleman passed on greetings to the other 12 gentlemen, after the second meeting he passed on greetings to 11 gentlemen, ... after the second-to-last meeting he passed on greetings to 1 gentleman, and after the last meeting he passed no greetings. In total, each member passed on $12+11+...+1+0=78$ greetings. This means that all 14 gentlemen passed on $14 + 78 = 1092$ greetings.

Problem 7. (12 credits) Let $1 = d_1 < d_2 < ... < d_k = n$ ($“d”$ are divisors of the natural number n).

It turned out that $d_2^2 + d_3^2 = n + 4$. Find n.

Answer: 16

Solution: If n is odd, all the divisors are odd, and the sum $d_2^2 + d_3^2$ is even, i.e. this leads to a contradiction. This means than n is even and $d_2 = 2$, $d_3^2 = n + 4 - d_2^2$, therefore d_3 is even, $d_3 = 2x$. If *x* is not equal to 2, it must precede d_3 in the sequence of the divisors. Therefore, $d_3=4$ and $n=16$.

Problem 8. (12 credits) In the triangle ABC, which is not isosceles, ∠A = 40°. A point moved along the bisector of the angle A, reflected from the side BC, and stopped at the same distance from the points B and C. At what angle is the side BC visible from this point?

Additional information:The angle of incidence of the point equals the angle of its reflection. It is considered that the segment XY is visible from the point *T* at the angle *XTY.*

Answer: 140°

Solution: Let the point mentioned in the conditions moved to point P, and let the point Q be symmetrical to point P with respect to the line BC. According to the conditions, PB=PC, this means that QB=QC, i.e. the point Q lies on the perpendicular bisector of the line BC.

As we know, the angle bisector of the triangle and the perpendicular bisector to the opposite side intersect on the circumscribed circle of the triangle. This means that the points A, B, Q, and C lie on the same circle, and the required angle is equal to ∠*BPC* = ∠*BQC* = 180° – ∠*BAC* = 180° – 40° = 140°.

Note. The problem can be solved by calculating the values of the angles without additional operations.

Problem 9. (14 credits) A mathematician calculated the sum $9 + 99 + 999 + ... + \underbrace{99...99}_{2025\ \text{times}}$ and wrote it on the blackboard. How many times does the digit ‘1’ appear in this result?

Answer: 2021

Solution: After adding 1 to each term we get the following:

$$S + 2025 = 10 + 100 + ... + 1\underbrace{00...00}_{2025\ \text{times}} = \underbrace{11...11}_{2025\ \text{times}}0,$$

$$S = \underbrace{11...11}_{2025\ \text{times}}0 - 2025 = \underbrace{11...11}_{2021\ \text{times}}00000 + 11110 - 2025 = \underbrace{11...11}_{2021\ \text{times}}09085.$$

The digit ‘1’ appear in this result 2021 times.

Problem 10. (14 credits) A binary operation $*$ assigns to every ordered pair of numbers a,b a number denoted by a∗b, such that for all a and b the following conditions hold true:

(1) $(a * b) + (b * a) = 2$,

(2) $(a + 1) * b = (a * b) + 3$.

Find the value of 1*2023.

Answer: - 6065.

Solution: It follows from the first condition that (1*1)+(1*1)=2, therefore (1*1)=1. Next, we get the following: 1*2023=2-2023*1=2-2022*1-3=2-2021*1-3-2=...=2-1*1-3-2022=-6065.

Note. In particular, the following formula is applicable: a*b=3(a-b)+1.