Answers and solutions to problems of the Green Difficulty Level
MathCat

Problem 1. (5 credits) In the morning, Mike and Serge walked toward each other and met 30 minutes later exactly halfway between their houses. At lunchtime, they rode their bicycles toward each other and met 15 minutes later exactly halfway again. How many minutes will it take them to meet in the evening after leaving their houses if Mike rides his bicycle and Serge walks? (All speeds are constant; Serge's walking speed and Mike's cycling speed are always the same.)

Answer: 20

Solution: The conditions show that walking speeds of Mike and Serge and their cycling speeds are equal to each other, since they met halfway. Moreover, their cycling speeds are 30*15=2 times greater than their walking speeds. Therefore, in the evening, Serge will only travel a third of the way to meet Mike (since his speed is 3 times slower than their speed of approach), and he will spend 30*2/3=20 minutes doing so.

Problem 2. (5 credits) Five boys sitting at a round table were brought 25 little pies for breakfast. Each boy ate either twice as many pies or two fewer pies than his neighbor to the right. What is the greatest number of pies the boys could have eaten?

Answer: 24

Solution: Let's show that each boy ate an even number of pies (this implies that the boys couldn't have eaten all 25 pies, but at most 24). Indeed, someone had to eat twice as many pies as his neighbor to the right, and then everyone would eat an even number of pies, otherwise each boy would eat 2 pies fewer than his neighbor to the right, and in this cast the first boy would eat 2*5=10 pies fewer than himself. I.e. someone ate an even number of pies, and the rest did the same. The boys could have eaten 24 pies in the following way: 2, 4, 8, 6, 4.

Problem 3. (7 credits) John supposed that if any natural number is multiplied by 104 and 1 is added to the product, the result is divisible by 105. When testing this John's supposition with the number 23/23*104=2392, Sam found that the number 2392+1=2393 is not divisible by 105. What is the largest three-digit number that becomes divisible by 105 after John's operations?

Answer: 946

Solution: Let n be a three-digit number. After John's operations, the resulting number is $104n+1$. This number must be divisible by 105. Note that $104n+1=105n-n+1=105n+(1-n)$. Since $105n$ is divisible by 105, $104n+1$ is divisible by 105 only if $(1-n)$ is divisible by 105, and this means that $(n-1)$ is divisible by 105. Thus, $n=105k+1$, where k is an integer ranging from 1 to 9. Therefore, the largest appropriate three-digit number is 105*9+1=946.

Problem 4. (9 credits) Three numbers are written in a row on the blackboard. A pupil continues this row to the right in the following way: each new number is equal to the sum of the rightmost number and the third number from the right. For example, the fourth number is equal to the sum of the third number and the first number, the fifth number is equal to the sum of the fourth number and the second number, and so on. In this way, he added four numbers. The sum of all seven numbers is 224. Find the sum of the first, second, third, and seventh numbers.

Answer: 112

Solution: Here are the first seven numbers: a, b, c, $a+c$, $a+b+c$, $a+b+2c$, $2a+b+3c$. The sum of all these numbers is $6a+4b+8c=224$. The required sum is $a+b+c+(2a+b+3c)=3a+2b+4c$, which is half the sum of all the numbers on the blackboard. Therefore, the answer is 224/2=112.

Problem 5. (10 credits) Jack wrote k natural numbers, the sum of which was 92. Jill calculated the sum of each number's digits and wrote those sums in the second row. Every number in Jill's sequence (except for the first one) was the same natural number greater than the previous one. What would be the largest k value in that case?

Answer: 10

Solution: We can see that the numbers in Jill's row differ by at least 1. Let's write out the smallest numbers ($nmin$) with the given sum of digits ($S(n)$):

$S(n)$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
n min	1	2	3	4	5	6	7	8	9	19	29	39	49	59	69	79	89	99

The sum of 11 numbers in the second row is 1+2+…+19+29=93. Therefore, $k<11$, i.e. $k=10$, and the sum is 2+…+19+29=92.

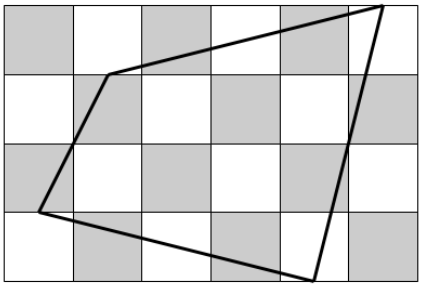
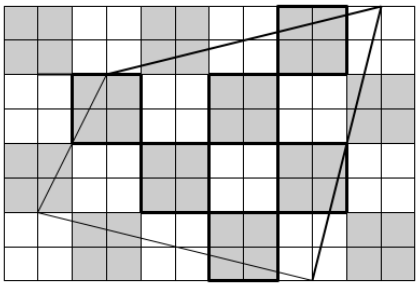


Figure 1

Problem 6. (10 credits) The vertices of the quadrilateral lie at the midpoints of sides of the squares, as shown in Figure 1. Find the grey area inside the quadrilateral if the area of one square is 1 square unit.
Answer: 6 square units
Solution: If we divide each square into 4 smaller squares, we can see that two of the highlighted gray cells are contained inside the quadrilateral completely, and internal parts of the remaining highlighted cells are supplemented by internal parts of other gray cells to make them whole.



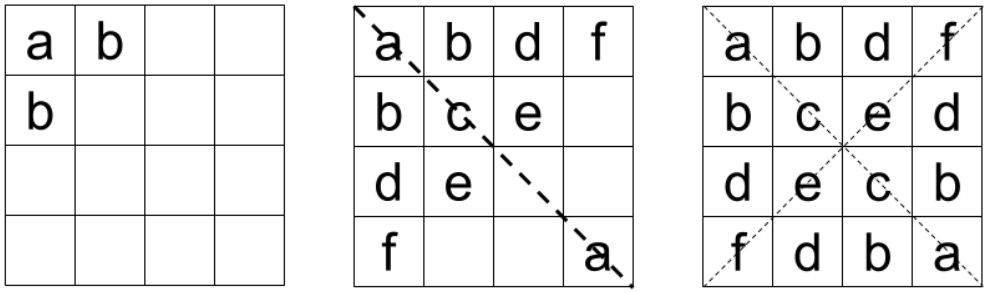
Problem 7. (12 credits) In a large family, each child was asked, 'How many brothers do you have?' The answer of each child was a natural number, and the sum of all those numbers was 44. How many children were in the family if all the children answered correctly?
Answer: 12 or 23
Solution: Let m be the number of boys and d be the number of girls; in this case, the total number of children is $n=m+d$. Each boy said that he has $m-1$ brothers, and each girl said that she has m brothers. The sum of the answers is: $m(m-1)+d\cdot m=m(m-1+d)=m(n-1)=44$. Since m and $n-1$ are natural numbers, m is a divisor of 44. 44 has the following divisors: 1, 2, 4, 11, 22, 44. Since the number of girls is non-negative, $n-1 \geq m-1$, i.e. $44/m \geq m-1$, therefore $m(m-1)\leq 44$. This inequality is satisfied only for $m=1$, $m=2$ and $m=4$. But if $m=1$, the boy would answer '0', which is not a natural number, therefore $m\geq 2$. In this case, $n=44/m+1$: for $m=2$ we get $n=23$, for $m=4$ we get $n=12$. Thus, the possible number of children in the family is 12 or 23.

Problem 8. (13 credits) All faces of seven dice are marked with all natural numbers from 1 to 42 (one number on each face, no duplicates). Before rolling the dice, Ann can wave her magic wand, causing each dice to come up the largest of the six numbers on its faces. What sum of numbers is guaranteed for Ann if she rolls all the dice?
Answer: 168
Solution: Let M_i be the largest number on the i-th dice (i=1, …, 7). Then we arrange the dice so that $M\_1 \leq M\_2 \leq \ldots \leq M\_7$. Since all seven dice together contain all the numbers from 1 to 42, the first dice contains 6 different numbers not exceeding M_1, i.e. $M\_1 \geq 6$. The first two dice contain 12 different numbers not exceeding M_2, i.e. $M\_2 \geq 12$, Similarly we get $M\_3 \geq 18$, $M\_4 \geq 24$, $M\_5 \geq 30$, $M\_6 \geq 36$ and $M\_7 \geq 42$. Consequently, the sum of the largest numbers on the dice is not less than 6+12+18+24+30+36+42=168. This sum is achievable, for example, if the numbers are distributed among the dice as follows: the first dice is marked with numbers from 1 to 6, the second one is marked with numbers from 7 to 12, …, the last dice is marked with numbers from 37 to 42, and we see that the sum is exactly 168.

Problem 9. (14 credits) In an acute triangle ABC, a bisector and an altitude are drawn from each angle. The angle between the bisector and the altitude is 8° for the angle A and 13° for the angle B. Find the angle between the bisector and the altitude drawn from the angle C.
Answer: 5 or 21
Solution: The angle between the bisector and the altitude drawn from any angle of a triangle is equal to half the absolute value of the difference between the other two angles of the triangle. Let us denote the angles between the bisector and the altitude drawn from the angles A, B, and C as α, β, and γ, respectively. This means that $\alpha=|B-C|/2=8°$, $\beta=|A-C|/2=13°$, $\gamma=|A-B|/2$. Note that γ can be equal to either the sum of α and β or

their difference: γ=α+β=21° or γ=|α−β|=5°. Thus, the angle between the bisector and the altitude of the angle *C* is equal to 21° or 5°.

Problem 10. (15 credits) Lucy colors all the cells of a 5x5 square board black and white. She likes the coloring if each cell has an even number of black adjacent cells. How many different colorings might Lucy like?
Answer: 32
Solution: Let's choose the topmost row in the square and color all the cells in this row black and white in random order. This can be done in 2*2*2*2*2=32 ways. Next, observe that the color of each cell in the rows below is uniquely determined, since for every cell in the preceding row all of its neighboring cells except one have already been colored. Thus, the coloring of the top row determines the coloring of all the other rows. The problem condition is automatically satisfied for all rows except the bottom one.



Let's show that the problem condition is satisfied for the bottom row as well. To do this, we'll use a 4x4 square board as an example to prove that any coloring of cells satisfying the required condition is centrally symmetric about the center of the square. Let's color the upper left corner cell color *a* (white or black; the exact color is irrelevant). We can see that the cells adjacent to this cell are colored the same color *b* (*from here on, we'll use different letters to denote colours, and different letters may denote the same or different colors; it's only important that identical letters denote the same color*). Let's denote *c* the color of the cell diagonally across from the cell colored *a*. Cells colored *b* have two neighboring cells colored *a* and *c*, and the third neighboring cells will be colored the same color *d*. A cell colored *c* has two neighboring cells of the same color (*b*), and two remaining neighboring cells will be colored the same color *e*. Continuing our reasoning, we find that two corner squares are also colored the same color *f* and that this coloring is symmetric about the diagonal *a-a* (shown by the dotted line in the figure). Now, starting the same reasoning with the upper right corner square, we find that the coloring is symmetric about the diagonal *f-f*, i.e. it is centrally symmetric about the center of the square. For squares of other sizes, the reasoning is similar.
To conclude the proof, we would like to note that since the condition is satisfied for the first two rows, it is also satisfied for the last two rows, including the bottom one.